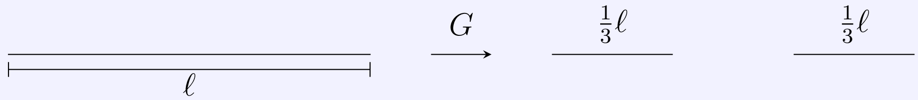
Grade 11/12 Math Circles
March 8 2023
Dynamical Systems and Fractals - Solutions

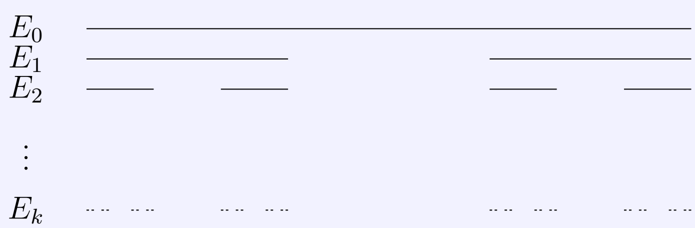
Exercise Solutions

Exercise 1

Consider the following generator, G.



which acts by removing the middle third of line segments. Repeated application of G results in the following fractal set, referred to as the Cantor set.



Determine an appropriate value of r and use the scaling relation to find the fractal dimension, D, of the Cantor set.

Letting $r = \frac{1}{3}$ we see that if it takes $N(\epsilon)$ measuring sticks of length ϵ (or ϵ-tiles if you prefer) to cover the Cantor set, then it will take $N\left(\frac{1}{3}\epsilon\right) = 2N(\epsilon)$ measuring sticks of length $\frac{1}{3}\epsilon$ to cover the Cantor set.

Putting this into the scaling relation we get

$$N\left(\frac{1}{3}\epsilon\right) = 2N(\epsilon) = N(\epsilon)\left(\frac{1}{3}\right)^{-D}.$$

Solving for D yields $D = \frac{\log(2)}{\log(3)} \approx 0.63$. The Cantor set is somewhere between zero-dimensional and one-dimensional.

Exercise 2

Consider the linear function $f(x) = ax + b$. Show that when $0 < a < 1$, $f(x)$ is a contraction mapping on the domain $[0, 1]$. Determine the contraction factor of f.

To show that f is a contraction mapping, consider x and $y \in [0, 1]$. We see that

$$\begin{aligned} |f(x) - f(y)| &= |ax + b - (ay + b)| \\ &= |ax - ay + b - b| \\ &= |ax - ay| \\ &= a|x - y| \\ &\leq a|x - y|. \end{aligned}$$

Since $0 < a < 1$, f is a contraction mapping. The contraction factor of f is a.

Problem Set Solutions

1. Consider the logistic function $f(x) = rx(1 - x)$ where $0 < r \leq 4$. In the lesson we saw (by looking at a plot of the iterates) that when $r > 3$ this function has a two-cycle. Now, let's show it algebraically. Last week we learned that we can solve for the period two points of $f(x)$ by solving the expression $f^{[2]}(\bar{x}) = \bar{x}$, however as $f(x)$ gets more complicated this can leave us with some messy equations to solve. In this question we will work through an easier way to solve for the two-cycle of $f(x)$.

   (a) Let $\{p_1, p_2\}$ be the two-cycle of $f(x)$. In order for this to be a two-cycle we must have that $f(p_1) = p_2$ and $f(p_2) = p_1$. Use this fact to write down two expressions relating p_1 and p_2.

   (b) Now subtract the two expressions you found in (a) and use the fact that $p_1 \neq p_2$ to simplify the resulting expression. You should end up with an expression which is linear in both p_1

and p_2.

(c) Finally, substitute this expression back into one of the expressions you found in (a) to solve for either p_1 or p_2. Use this result to show that $f(x)$ only has a (real-valued) two-cycle when $r > 3$.

Solution:

(a) Using $f(p_1) = p_2$ and $f(p_2) = p_1$ we have the following two expressions

$$rp_1(1 - p_1) = p_2$$
$$rp_2(1 - p_2) = p_1$$

which relate p_1 and p_2.

(b) Subtracting the two expressions from (a) gives

$$\begin{aligned} rp_1(1 - p_1) - rp_2(1 - p_2) &= p_2 - p_1 \\ r(p_1 - p_2) - r(p_1^2 - p_2^2) &= p_2 - p_1 \\ r(p_1 - p_2) - r(p_1 - p_2)(p_1 + p_2) &= p_2 - p_1. \end{aligned}$$

Since $p_2 \neq p_1$ (by the definition of a two-cycle) we can divide both sides by $p_1 - p_2$, resulting in

$$\begin{aligned} r - r(p_1 + p_2) &= -1 \\ p_1 + p_2 &= \frac{1 + r}{r} \\ p_1 &= \frac{1 + r}{r} - p_2. \end{aligned}$$

(c) Finally, we substitute our result from (b) back into one of our expressions from (a) to solve for p_1 or p_2. Since p_1 and p_2 are interchangeable in our initial formulation it doesn't matter which one we solve for.

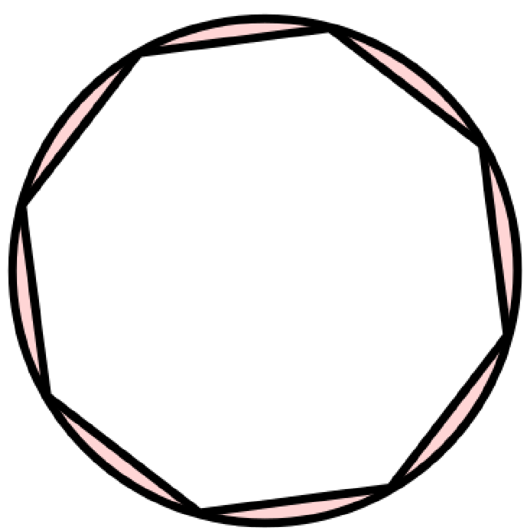
$$\begin{aligned} rp_2(1-p_2) &= \frac{1+r}{r} - p_2 \\ rp_2 - rp_2^2 &= \frac{1+r}{r} - p_2 \\ rp_2^2 - (r+1)p_2 + \frac{1+r}{r} &= 0. \end{aligned}$$

Using the quadratic formula we get

$$\begin{aligned} p_2 &= \frac{r+1}{2r} \pm \frac{\sqrt{(r+1)^2 - 4r\frac{1+r}{r}}}{2r} \\ &= \frac{r+1}{2r} \pm \frac{\sqrt{(r+1)(r-3)}}{2r}. \end{aligned}$$

Since $r > 0$, this has two distinct (real) solutions when $r > 3$. Thus, we have a two-cycle when $r > 3$.

2. Consider a circle C which has radius 1. Now consider inscribing C with a regular polygon P_n which has 2^n equal sides, as shown in the figure below. The idea is that we can consider the length (L_n) of the perimeter of P_n as an approximation for the circumference ($L = 2\pi$) of the circle C.



(a) Write down an expression for L_n (the length of the perimeter of P_n).

(b) **CHALLENGE** (You will need to be familiar with limits in order to solve this next part.) Show that $\lim_{n\to\infty} L_n = L = 2\pi$.

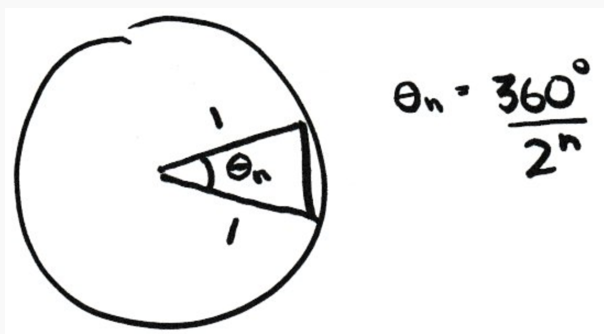
Hint: You may work with angles in either degrees or radians (if you are familiar with radians). You will need to use the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ *(when* x *is in radians) or that* $\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\pi}{180}$ *(when* x *is in degrees).*

Solution: Working with angles in degrees (solution using radians is very similar):

(a) To start, we need the length of each side of P_n, which is given by

$$2 \cdot \sin\left(\frac{\theta_n}{2}\right) = 2 \cdot \sin\left(\frac{360^\circ}{2^{n+1}}\right)$$

as seen on the following figure.



Since P_n has 2^n sides, the length of its perimeter is given by

$$\begin{aligned} L_n &= 2^n \cdot 2 \cdot \sin\left(\frac{360^\circ}{2^{n+1}}\right) \\ &= 2^{n+1} \cdot \sin\left(\frac{360^\circ}{2^{n+1}}\right). \end{aligned}$$

(b)

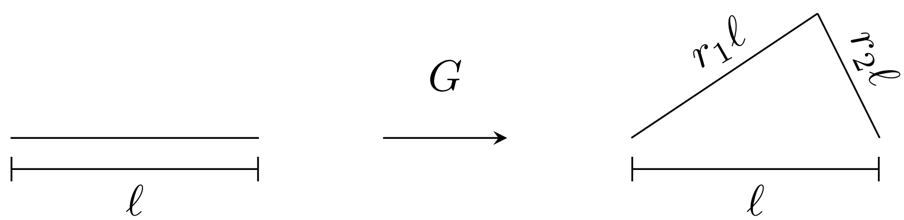
$$\begin{aligned} L = \lim_{n\to\infty} L_n &= \lim_{n\to\infty} 2^{n+1} \cdot \sin\left(\frac{360^\circ}{2^{n+1}}\right) \\ &= \lim_{n\to\infty} \frac{\sin\left(\frac{360^\circ}{2^{n+1}}\right)}{\frac{1}{2^{n+1}}} \end{aligned}$$

Now, let $x_n = \frac{360°}{2^{n+1}}$. As $n \to \infty$, $x_n \to 0$.

$$\begin{aligned}\lim_{n\to\infty} L_n &= \lim_{x_n\to 0} \frac{\sin(x_n)}{\frac{x_n}{360°}} \\ &= 360° \lim_{x_n\to 0} \frac{\sin(x_n)}{x_n} \\ &= 360° \cdot \frac{\pi}{180°} \\ &= 2\pi\end{aligned}$$

which gives the desired result.

3. Consider the generator G sketched below:



where $0 < r_1 < 1, 0 < r_2 < 1$ and $1 < r_1 + r_2 < 2$.

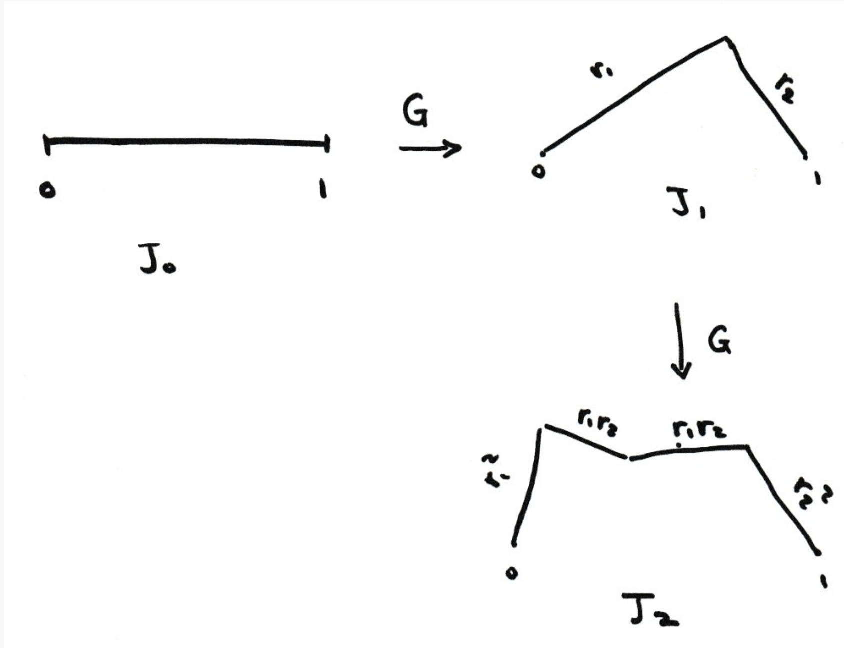
(a) Starting with the set $J_0 = [0, 1]$, sketch $J_1 = G(J_0)$ and $J_2 = G(J_1)$.

(b) What is the length of J_1 (L_1)? Of J_2 (L_2)? In general, can you find an expression for the length of $J_n = G^n(J_0)$?

(c) What do you expect to happen to the length of J_n as n gets infinitely large (i.e. as the set J_n approaches the attractor)?

Solution:

(a) J_1 and J_2 are as follows



(b) The length of J_1 is $L_1 = r_1 + r_2$.

The length of J_2 is $L_2 = r_1^2 + 2r_1r_2 + r^2 = (r_1 + r_2)^2$.

In general, G scales the length of each line segment by a factor of $r_1 + r_2$ so we can write $L_n = (r_1 + r_2)^n$.

(c) Since $r_1 + r_2 > 1$, $\lim_{n\to\infty} L_n = \lim_{n\to\infty}(r_1 + r_2)^n = \infty$. In other words, as n gets infinitely large, the length of J_n will approach infinity (meaning that the attractor has infinite length).

4. Consider the following two function iterated function system (IFS) on $[0, 1]$,

$$f_1(x) = \frac{1}{5}x, \quad f_2(x) = \frac{1}{5}x + \frac{4}{5}.$$

(a) Let $I_0 = [0, 1]$ and $I_1 = F(I_0)$ where F is the parallel IFS operator composed of the two functions f_1 and f_2. Sketch I_1 on the real number line.

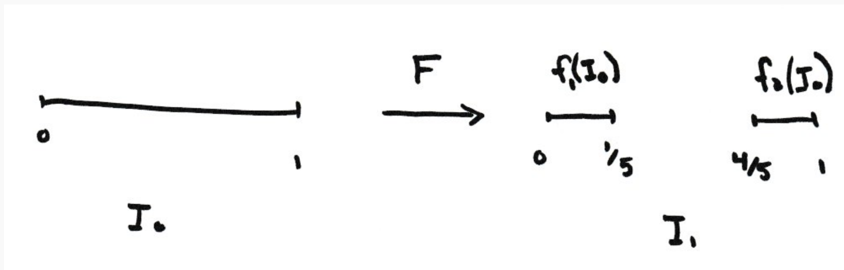
(b) Let $I_2 = F(I_1)$. Sketch I_2 on the real number line.

(c) Let I denote the limiting set (or attractor) of this IFS. Use the scaling relation to determine the fractal dimension D of I.

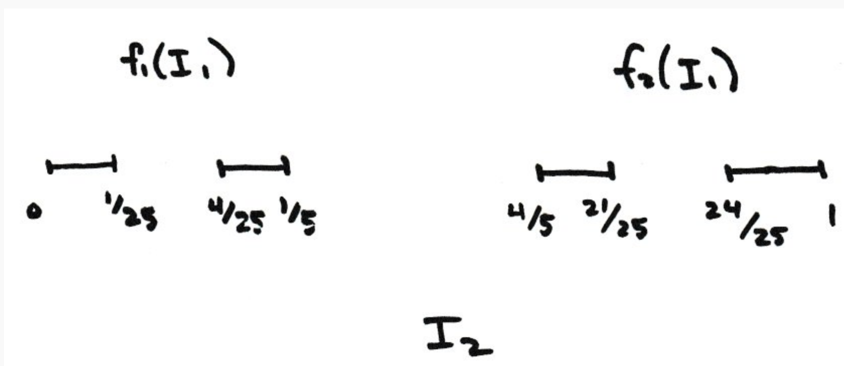
Hint: D will be a ratio of two logarithms.

Solution:

(a) I_1 is as follows



(b) I_2 is as follows



(c) Letting $r = \frac{1}{5}$ we see that $N(r\epsilon) = 2N(\epsilon)$ (one measuring stick of length one, two of length $\frac{1}{5}$, four of length $\frac{1}{25}$, etc...). Putting this into the scaling relation we get

$$N\left(\frac{1}{5}\epsilon\right) = 2N(\epsilon) = N(\epsilon)\left(\frac{1}{5}\right)^{-D}$$

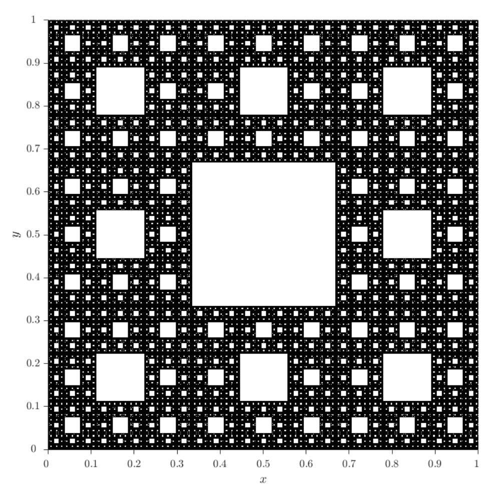
which implies

$$\begin{aligned} 2N(\epsilon) &= N(\epsilon)\left(\frac{1}{5}\right)^{D} \\ 2 &= 5^D \\ D &= \frac{\log(2)}{\log(5)} \approx 0.43. \end{aligned}$$

5. Show that the function $f(x) = x^2$ is a contraction mapping on the domain $[0, \frac{1}{4}]$. Determine the contraction factor of f.

> *Solution*: Let $x, y \in \left[0, \frac{1}{4}\right]$. Then
>
> $$\begin{aligned}|f(x) - f(y)| &= |x^2 - y^2| \\ &= |x + y||x - y| \\ &\leq \frac{1}{2}|x - y|.\end{aligned}$$
>
> Therefore f is a contraction mapping with contraction factor $\frac{1}{2}$ on the domain $\left[0, \frac{1}{4}\right]$.

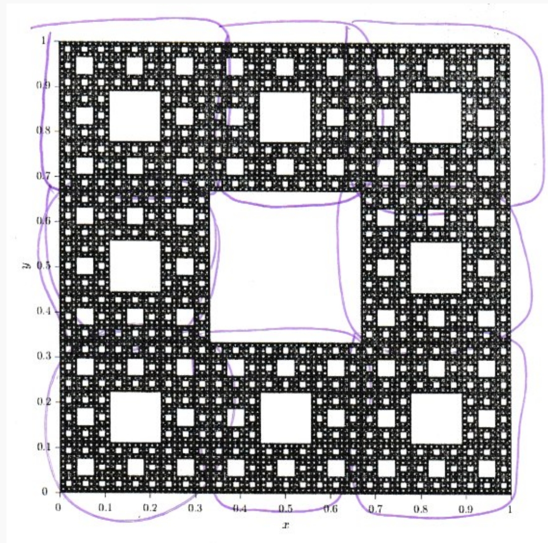
6. Consider the image of the Sierpinski carpet, S, shown below. The Sierpinski carpet is a self-similar fractal which means that is a union of contracted copies of itself.



(a) Show (by circling them on the figure) that S is made up of eight contracted copies of itself. What is the contraction factor of these copies?

(b) Determine the similarity dimension of S.

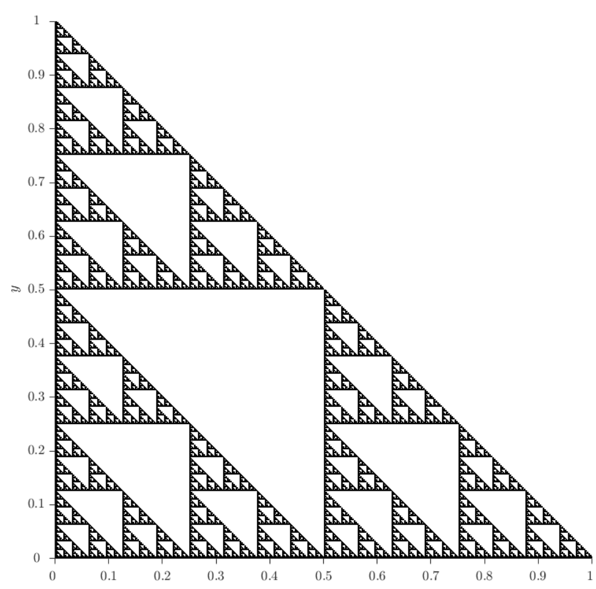
> *Solution*:
>
> (a) We see that there are eight contracted copies of S, as shown on the following figure. We can also see from the figure that each copy of S is scaled down by $\frac{1}{3}$, or has a contraction factor of $\frac{1}{3}$.



(b) Since S is made up of eight copies of itself, each scaled by a factor of $\frac{1}{3}$, the similarity dimension of S is

$$D = \frac{\log(8)}{\log(3)} \approx 1.89.$$

7. Consider the image of the modified Sierpinski triangle, S, shown below.



(a) Show (by circling them on the figure) that S is made up of three contracted copies of itself.

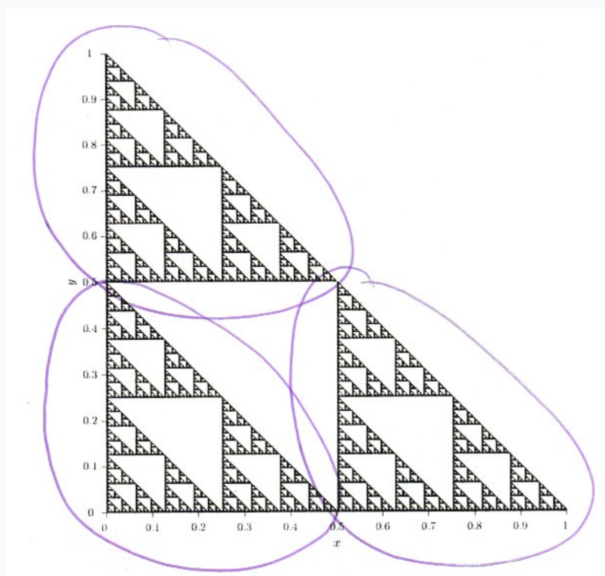
(b) Imagine starting with a right triangle, S_0, which has vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Describe (in terms of contraction factors, translations, rotations, etc...) the three map IFS which you could use to construct S from S_0.

(c) Determine the similarity dimension of S.

(d) **CHALLENGE** Describe a fourth map which could be added to the IFS you found in (b) so that the attractor of the IFS is a solid triangular region.

Solution:

(a) We can see from the figure below that S is made up of three contracted copies of itself.



(b) We can see from the figure that S is made up of three scaled copies of itself, each contracted by a factor of $\frac{1}{2}$.

The first map simply scales S_0 by $\frac{1}{2}$, resulting in the triangle in the bottom left corner. The second map scales S_0 by $\frac{1}{2}$ and translates it to the right by $\frac{1}{2}$, resulting in the triangle in the bottom right corner. Finally, the third map scales S_0 by $\frac{1}{2}$ and translates it upwards by $\frac{1}{2}$ to form the triangle in the top corner.

(c) The similarity dimension of S is given by

$$D = \frac{\log(3)}{\log(2)} \approx 1.56.$$

(d) If we want the attractor of the IFS to be a solid triangular region, we need to add a fourth map which will fill up the triangular gap in the middle. We can do this by defining a map which scales S_0 by $\frac{1}{2}$, rotates it by $180°$ and translates it by $\frac{1}{2}$ up and to the right.